Supersymmetric and Shape-Invariant Generalization for Nonresonant and Intensity-Dependent Jaynes-Cummings Systems

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Abstract

A class of shape-invariant bound-state problems which represent transition in a two-level system introduced earlier are generalized to include arbitrary energy splittings between the two levels as well as intensity-dependent interactions. We show that the couple-channel Hamiltonians obtained correspond to the generalizations of the nonresonant and intensity-dependent nonresonant Jaynes-Cummings Hamiltonians, widely used in quantized theories of laser. In this general context, we determine the eigenstates, eigenvalues, the time

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evolution matrix and the population inversion matrix factor.

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I. INTRODUCTION

The integrability condition called shape-invariance originates in supersymmetric quantum mechanics [1,2]. The separable positive-definite Hamiltonian $\hat{H}_1 = \hat{A}^\dagger \hat{A}$ is called shape-invariant if the condition

$$\hat{A}(a_1)\hat{A}^\dagger(a_1) = \hat{A}^\dagger(a_2)\hat{A}(a_2) + R(a_1)\,, \tag{1.1}$$

is satisfied [3]. In this equation a_1 and a_2 represent parameters of the Hamiltonian. The parameter a_2 is a function of a_1 and the remainder $R(a_1)$ is independent of the dynamical variables such as position and momentum. Even though not all exactly-solvable problems are shape-invariant [4], shape invariance, especially in its algebraic formulation [5–7], has proven to be a powerful technique to study exactly-solvable systems.

In a previous paper [8] we used shape-invariance to calculate the energy eigenvalues and eigenfunctions for the Hamiltonian

$$\mathbf{\hat{H}} = \hat{A}^\dagger \hat{A} + \frac{1}{2}\left[\hat{A}, \hat{A}^\dagger\right](\hat{\sigma}_3 + 1) + \sqrt{\hbar\Omega}\left(\hat{\sigma}_+ \hat{A} + \hat{\sigma}_- \hat{A}^\dagger\right)\,, \tag{1.2}$$

where

$$\hat{\sigma}_\pm = \frac{1}{2}\left(\hat{\sigma}_1 \pm i\hat{\sigma}_2\right)\,, \tag{1.3}$$

and $\hat{\sigma}_i$, with $i = 1$, 2, and 3, are the Pauli matrices.

This is a generalization of the Jaynes-Cummings Hamiltonian [9]. A different, but related problem was considered in Ref. [10]. Our goal in this paper is to study further generalizations of the Jaynes-Cummings Hamiltonian, first by introducing a term proportional to σ_3 with an arbitrary coefficient (the so-called nonresonant limit) and then by taking into account the dependence of the coupling on the intensity of the field (the so-called intensity-dependent nonresonant limit). In addition to the energy levels we study the time evolution and the population inversion matrix factor.

Introducing the similarity transformation that replaces a_1 with a_2 in a given operator

$$\hat{T}(a_1)\,\hat{O}(a_1)\,\hat{T}^\dagger(a_1) = \hat{O}(a_2) \tag{1.4}$$

and the operators

$$\hat{B}_+ = \hat{A}^\dagger(a_1)\hat{T}(a_1) \tag{1.5}$$

$$\hat{B}_- = \hat{B}_+^\dagger = \hat{T}^\dagger(a_1)\hat{A}(a_1)\,, \tag{1.6}$$

the condition of Eq. (1.1) can be written as a commutator [5]

$$[\hat{B}_-, \hat{B}_+] = \hat{T}^\dagger(a_1)R(a_1)\hat{T}(a_1) \equiv R(a_0)\,, \tag{1.7}$$

where we used the identity

$$R(a_n) = \hat{T}(a_1)\,R(a_{n-1})\,\hat{T}^\dagger(a_1)\,, \tag{1.8}$$

valid for any n. The ground state of the Hamiltonian $\hat{H}_1 = \hat{A}^\dagger \hat{A} = \hat{B}_+ \hat{B}_-$ satisfies the condition

$$\hat{A} \mid \psi_0 \rangle = 0 = \hat{B}_- \mid \psi_0 \rangle \, ; \tag{1.9}$$

and the unnormalized n-th excited state is given by

$$\mid \psi_n \rangle \sim \left(\hat{B}_+ \right)^n \mid \psi_0 \rangle \tag{1.10}$$

with the eigenvalue

$$\mathcal{E}_n = \sum_{k=1}^{n} R(a_k) \, . \tag{1.11}$$

We note that the Hamiltonian of Eq. (1.2) can also be written as

$$\hat{\mathbf{H}} = \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \hat{\mathbf{h}}_\pm \begin{bmatrix} \hat{T}^\dagger & 0 \\ 0 & \pm 1 \end{bmatrix} , \tag{1.12}$$

where

$$\hat{\mathbf{h}}_\pm = \hat{B}_+ \hat{B}_- + \frac{1}{2} R(a_0) \left(\hat{\sigma}_3 + 1 \right) \pm \sqrt{\hbar \Omega} \left(\hat{\sigma}_+ \hat{B}_- + \hat{\sigma}_- \hat{B}_+ \right) \, . \tag{1.13}$$

II. THE GENERALIZED NONRESONANT JAYNES-CUMMINGS HAMILTONIAN

The standard Jaynes-Cummings model, normally used in quantum optics, idealizes the interaction of matter with electromagnetic radiation by a simple Hamiltonian of a two-level atom coupled to a single bosonic mode [11–16]. This Hamiltonian has a fundamental importance to the field of quantum optics and it is a central ingredient in the quantized description of any optical system involving the interaction between light and atoms. The Jaynes-Cummings Hamiltonian defines a *molecule*, a composite system formed from the coupling of a two-state system and a quantized harmonic oscillator. In this case, its nonresonant expression can be written as

$$\hat{\mathbf{H}} = \hat{A}^\dagger \hat{A} + \frac{1}{2} \left[\hat{A}, \hat{A}^\dagger \right] \left(\hat{\sigma}_3 + 1 \right) + \alpha \left(\hat{\sigma}_+ \hat{A} + \hat{\sigma}_- \hat{A}^\dagger \right) + \hbar \Delta \, \hat{\sigma}_3 \, , \tag{2.1}$$

where α is a constant related with the coupling strength and Δ is a constant related with the detuning of the system.

However, the harmonic oscillator systems, used in this context, is only the simplest example of supersymmetric and shape-invariant potential. Our goal here is to generalize that Hamiltonian for all supersymmetric and shape-invariant systems. With this purpose and following Ref. [8] we introduce the operator

$$\hat{\mathbf{S}} = \hat{\sigma}_+ \hat{A} + \hat{\sigma}_- \hat{A}^\dagger \, , \tag{2.2}$$

where the operators $\hat{A}$ and $\hat{A}^\dagger$ satisfy the shape invariance condition of Eq. (1.1). Using this definition we can decompose the nonresonant Jaynes-Cummings Hamiltonian in the form

$$\hat{\mathbf{H}} = \hat{\mathbf{H}}_o + \hat{\mathbf{H}}_{int}\,, \tag{2.3}$$

where

$$\hat{\mathbf{H}}_o = \hat{\mathbf{S}}^2\,, \tag{2.4a}$$
$$\hat{\mathbf{H}}_{int} = \alpha\,\hat{\mathbf{S}} + \hbar\Delta\,\hat{\sigma}_3\,. \tag{2.4b}$$

First, we search for the eigenstates of $\hat{\mathbf{S}}^2$. In this case it is more convenient to work with its B-operator expression, which can be written as [8]

$$\hat{\mathbf{S}}^2 = \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} \hat{B}_-\hat{B}_+ & 0 \\ 0 & \hat{B}_+\hat{B}_- \end{bmatrix} \begin{bmatrix} \hat{T}^\dagger & 0 \\ 0 & \pm 1 \end{bmatrix} \equiv \begin{bmatrix} \hat{H}_2 & 0 \\ 0 & \hat{H}_1 \end{bmatrix}, \tag{2.5}$$

where $\hat{H}_2 = \hat{T}\hat{B}_-\hat{B}_+\hat{T}^\dagger$. Note the freedom of sign choice in Eq. (2.5), which results in two possible decompositions of $\hat{\mathbf{S}}^2$. Next, we introduce the states

$$\mid \Psi^{(\pm)}\rangle = \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} C_m^{(\pm)} \mid m\rangle \\ C_n^{(\pm)} \mid n\rangle \end{bmatrix}, \tag{2.6}$$

where $C_{m,n}^{(\pm)} \equiv C_{m,n}^{(\pm)}\left[R(a_1), R(a_2), R(a_3), \ldots\right]$ are auxiliary coefficients and, $\mid m\rangle$ and $\mid n\rangle$ are the abbreviated notation for the states $\mid \psi_m\rangle$ and $\mid \psi_n\rangle$ of Eq. (1.10). Using Eqs. (1.7), (2.5) and (2.6), the commutation between $\hat{H}_1$ and a function of $R(a_k)$, and the $\hat{T}$-operator unitary condition, we get

$$\begin{aligned} \hat{\mathbf{S}}^2 \mid \Psi^{(\pm)}\rangle &= \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} \hat{B}_+\hat{B}_- + R(a_0) & 0 \\ 0 & \hat{B}_+\hat{B}_- \end{bmatrix} \begin{bmatrix} C_m^{(\pm)} \mid m\rangle \\ C_n^{(\pm)} \mid n\rangle \end{bmatrix} \\ &= \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} \mathcal{E}_m + R(a_0) & 0 \\ 0 & \mathcal{E}_n \end{bmatrix} \begin{bmatrix} C_m^{(\pm)} \mid m\rangle \\ C_n^{(\pm)} \mid n\rangle \end{bmatrix}. \end{aligned} \tag{2.7}$$

And using Eqs. (1.8) and (1.11) we can write

$$\begin{aligned} \hat{T}\left[\mathcal{E}_m + R(a_0)\right]\hat{T}^\dagger &= \hat{T}\left[R(a_1) + R(a_2) + \cdots + R(a_m) + R(a_0)\right]\hat{T}^\dagger \\ &= R(a_2) + R(a_3) + \cdots + R(a_{m+1}) + R(a_1) = \mathcal{E}_{m+1}\,. \end{aligned} \tag{2.8}$$

Hence the states

$$\mid \Psi_m^{(\pm)}\rangle = \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} C_m^{(\pm)} \mid m\rangle \\ C_{m+1}^{(\pm)} \mid m+1\rangle \end{bmatrix}, \qquad m = 0, 1, 2, \cdots \tag{2.9}$$

are the normalized eigenstates of the operator $\hat{\mathbf{S}}^2$

$$\hat{\mathbf{S}}^2 \mid \Psi_m^{(\pm)}\rangle = \mathcal{E}_{m+1} \mid \Psi_m^{(\pm)}\rangle\,. \tag{2.10}$$

We observe that the orthonormality of the wavefunctions imply in the following relations among the C's

$$\langle \Psi_m^{(\pm)} \mid \Psi_m^{(\pm)} \rangle = \left[C_m^{(\pm)}\right]^2 + \left[C_{m+1}^{(\pm)}\right]^2 = 1 \tag{2.11a}$$

$$\langle \Psi_m^{(\mp)} \mid \Psi_m^{(\pm)} \rangle = C_m^{(\pm)} C_m^{(\mp)} - C_{m+1}^{(\pm)} C_{m+1}^{(\mp)} = 0\,. \tag{2.11b}$$

Since $\hat{\mathbf{S}}^2$ and $\hat{\mathbf{H}}_{int}$ commute then it is possible to find a common set of eigenstates. We can use this fact to determine the eigenvalues of $\hat{\mathbf{H}}_{int}$ and the relations among the C's coefficients. For that we need to calculate

$$\hat{\mathbf{H}}_{int} \mid \Psi_m^{(\pm)} \rangle = \lambda_m^{(\pm)} \mid \Psi_m^{(\pm)} \rangle\,, \tag{2.12}$$

where $\lambda_m^{(\pm)}$ are the eigenvalues to be determined. Using Eqs. (2.2), (2.4) and (2.9), the last eigenvalue equation can be rewritten in a matrix form as

$$\alpha \begin{bmatrix} \beta & \hat{T}\hat{B}_- \\ \hat{B}_+\hat{T}^\dagger & -\beta \end{bmatrix} \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} C_m^{(\pm)} \mid m\rangle \\ C_{m+1}^{(\pm)} \mid m+1\rangle \end{bmatrix} = \lambda_m^{(\pm)} \begin{bmatrix} C_m^{(\pm)} \mid m\rangle \\ C_{m+1}^{(\pm)} \mid m+1\rangle \end{bmatrix}, \tag{2.13}$$

where $\beta = \hbar\Delta/\alpha$. Since the C's coefficients commute with the $\hat{A}$ or $\hat{A}^\dagger$ operators, then the last matrix equation permits to obtain the following equations

$$\left[\alpha\beta - \lambda_m^{(\pm)}\right] \left(\hat{T} C_m^{(\pm)} \hat{T}^\dagger\right) \hat{T} \mid m\rangle \pm \alpha\, C_{m+1}^{(\pm)} \hat{T}\hat{B}_- \mid m+1\rangle = 0 \tag{2.14a}$$

$$\alpha \left(\hat{T} C_m^{(\pm)} \hat{T}^\dagger\right) \hat{B}_+ \mid m\rangle \mp \left[\alpha\beta + \lambda_m^{(\pm)}\right] C_{m+1}^{(\pm)} \mid m+1\rangle = 0\,. \tag{2.14b}$$

Introducing the operator [7]

$$\hat{Q}^\dagger = \left(\hat{B}_+\hat{B}_-\right)^{-1/2} \hat{B}_+ \tag{2.15}$$

one can write the normalized eigenstate of $\hat{H}_1$ as

$$\mid m\rangle = \left(\hat{Q}^\dagger\right)^m \mid 0\rangle\,, \tag{2.16}$$

and, with Eqs. (2.15) and (2.16) we can show that [8]

$$\hat{B}_+ \mid m\rangle = \sqrt{\mathcal{E}_{m+1}} \mid m+1\rangle\,, \tag{2.17a}$$

$$\hat{T}\hat{B}_- \mid m+1\rangle = \sqrt{\mathcal{E}_{m+1}}\, \hat{T} \mid m\rangle\,. \tag{2.17b}$$

Substituting Eqs. (2.17) into Eqs. (2.14) we have

$$\left\{\left[\alpha\beta - \lambda_m^{(\pm)}\right] \left(\hat{T} C_m^{(\pm)} \hat{T}^\dagger\right) \pm \alpha\sqrt{\mathcal{E}_{m+1}}\, C_{m+1}^{(\pm)}\right\} \hat{T} \mid m\rangle = 0 \tag{2.18a}$$

$$\left\{\alpha\sqrt{\mathcal{E}_{m+1}} \left(\hat{T} C_m^{(\pm)} \hat{T}^\dagger\right) \mp \left[\alpha\beta + \lambda_m^{(\pm)}\right] C_{m+1}^{(\pm)}\right\} \mid m+1\rangle = 0\,. \tag{2.18b}$$

From Eqs. (2.18) it follows that

$$\lambda_m^{(\pm)} = \pm\alpha\sqrt{\mathcal{E}_{m+1} + \beta^2}\,, \tag{2.19}$$

and

$$C_{m+1}^{(\pm)} = \left(\frac{\sqrt{\mathcal{E}_{m+1} + \beta^2} \mp \beta}{\sqrt{\mathcal{E}_{m+1}}} \right) \left(\hat{T} C_m^{(\pm)} \hat{T}^\dagger \right) . \tag{2.20}$$

Eqs. (2.11) and (2.20) imply that

$$C_{m+1}^{(\pm)} = C_m^{(\mp)} , \tag{2.21}$$

and the eigenstates and eigenvalues of the generalized nonresonant Jaynes-Cummings Hamiltonians can be written as

$$E_m^{(\pm)} = \mathcal{E}_{m+1} \pm \sqrt{\alpha^2 \, \mathcal{E}_{m+1} + \hbar^2 \Delta^2} , \tag{2.22}$$

and

$$\mid \Psi_m^{(\pm)} \rangle = \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} C_m^{(\pm)} \mid m \rangle \\ C_m^{(\mp)} \mid m+1 \rangle \end{bmatrix} , \qquad m = 0, 1, 2, \cdots \tag{2.23}$$

a) The Resonant Limit

From these general results we can verify two important and simple limiting cases. The first one corresponds to the resonant situation, for which $\Delta = 0$ $(\beta = 0)$. Using these conditions in Eqs. (2.20) and (2.22) and Eqs. (2.11) we get

$$E_m^{(\pm)} = \mathcal{E}_{m+1} \pm \sqrt{\alpha^2 \, \mathcal{E}_{m+1}} , \tag{2.24}$$

and

$$C_{m+1}^{(\pm)} = \hat{T} C_m^{(\pm)} \hat{T}^\dagger = C_m^{(\pm)} = \frac{1}{\sqrt{2}} . \tag{2.25}$$

Therefore the Jaynes-Cummings resonant eigenstate is given by

$$\mid \Psi_m^{(\pm)} \rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} \mid m \rangle \\ \mid m+1 \rangle \end{bmatrix} , \qquad m = 0, 1, 2, \cdots \tag{2.26}$$

These particular results are shown in the Ref. [8].

b) The Standard Jaynes-Cummings Limit

The second important limit corresponds to the standard Jaynes-Cummings Hamiltonian, related with the harmonic oscillator system. In this limit we have that $\hat{T} = \hat{T}^\dagger \longrightarrow 1$, $\hat{B}_- \longrightarrow \hat{a}$, $\hat{B}_+ \longrightarrow \hat{a}^\dagger$, $\Delta = \omega - \omega_o$ and $\mathcal{E}_{m+1} = (m+1)\hbar\omega$. Using these conditions in the Eqs. (2.20), (2.22) and Eqs. (2.11) we conclude that

$$E_m^{(\pm)} = (m+1)\hbar\omega \pm \sqrt{\alpha^2 \hbar\omega (m+1) + \hbar^2 (\omega - \omega_o)^2} , \tag{2.27}$$

and

$$C_{m+1}^{(\pm)} = \gamma_m^{(\pm)} C_m^{(\pm)} = C_m^{(\mp)} = \frac{1}{\sqrt{1+\left(\gamma_m^{(\mp)}\right)^2}}\,, \tag{2.28}$$

where

$$\gamma_m^{(\pm)} = \sqrt{1+\delta_m^2} \mp \delta_m\,, \tag{2.29a}$$

$$\delta_m = \frac{\hbar(\omega-\omega_o)}{\sqrt{(m+1)\alpha^2\hbar\omega}}\,. \tag{2.29b}$$

Therefore the standard Jaynes-Cummings eigenstate, written in a matrix form, is given by

$$\mid \Psi_m^{(\pm)}\rangle = \frac{1}{\sqrt{1+\left(\gamma_m^{(\pm)}\right)^2}} \begin{bmatrix} 1 & 0 \\ 0 & \pm\gamma_m^{(\pm)} \end{bmatrix} \begin{bmatrix} \mid m\rangle \\ \mid m+1\rangle \end{bmatrix}, \qquad m = 0,1,2,\cdots \tag{2.30}$$

These results are shown in many papers, in particular, in the Ref. [17].

III. THE TIME EVOLUTION OF THE NONRESONANT SYSTEM

To study the time-dependent Schrödinger equation for a Jaynes-Cummings system in nonresonant situation

$$i\hbar\frac{\partial}{\partial t} \mid \Psi(t)\rangle = \left(\hat{\mathbf{H}}_o + \hat{\mathbf{H}}_{int}\right) \mid \Psi(t)\rangle \tag{3.1}$$

we can write the wavefunction as

$$\mid \Psi(t)\rangle = \exp\left(-i\hat{\mathbf{H}}_o t/\hbar\right) \mid \Psi_i(t)\rangle\,, \tag{3.2}$$

and, by substituting this into Schrödinger equation and taking into account the commutation property between $\hat{\mathbf{H}}_o$ and $\hat{\mathbf{H}}_{int}$, we obtain

$$i\hbar\frac{\partial}{\partial t} \mid \Psi_i(t)\rangle = \hat{\mathbf{H}}_{int} \mid \Psi_i(t)\rangle\,. \tag{3.3}$$

We introduce the evolution matrix $\hat{\mathbf{U}}_i(t,0)$:

$$\mid \Psi_i(t)\rangle = \hat{\mathbf{U}}_i(t,0) \mid \Psi_i(0)\rangle\,. \tag{3.4}$$

which satisfies the equation

$$i\hbar\frac{\partial}{\partial t}\hat{\mathbf{U}}_i(t,0) = \hat{\mathbf{H}}_{int}\,\hat{\mathbf{U}}_i(t,0)\,, \tag{3.5}$$

that is, in matrix form, written as

$$i\hbar \begin{bmatrix} \hat{U}'_{11} & \hat{U}'_{12} \\ \hat{U}'_{21} & \hat{U}'_{22} \end{bmatrix} = \alpha \begin{bmatrix} \beta & \hat{T}\hat{B}_- \\ \hat{B}_+\hat{T}^\dagger & -\beta \end{bmatrix} \begin{bmatrix} \hat{U}_{11} & \hat{U}_{12} \\ \hat{U}_{21} & \hat{U}_{22} \end{bmatrix}\,, \tag{3.6}$$

where the primes denote the time derivative. One fast way to diagonalize the evolution matrix differential equation is by differentiating Eq. (3.5) with respect to time. We find

$$i\hbar\frac{\partial^2}{\partial t^2}\hat{\mathbf{U}}_i(t,0) = \hat{\mathbf{H}}_{int}\,\frac{\partial}{\partial t}\hat{\mathbf{U}}_i(t,0) = \frac{1}{i\hbar}\hat{\mathbf{H}}^2_{int}\hat{\mathbf{U}}_i(t,0)\,, \tag{3.7}$$

which can be written as

$$\begin{bmatrix}\hat{U}''_{11} & \hat{U}''_{12}\\ \hat{U}''_{21} & \hat{U}''_{22}\end{bmatrix} = -\begin{bmatrix}\hat{\omega}_1 & 0\\ 0 & \hat{\omega}_2\end{bmatrix}\begin{bmatrix}\hat{U}_{11} & \hat{U}_{12}\\ \hat{U}_{21} & \hat{U}_{22}\end{bmatrix}\,, \tag{3.8}$$

where

$$\hbar\hat{\omega}_1 = \alpha\sqrt{\hat{T}\hat{B}_-\hat{B}_+\hat{T}^\dagger + \beta^2} = \sqrt{\alpha^2\,\hat{H}_2 + (\hbar\Delta)^2}\,, \tag{3.9a}$$

$$\hbar\hat{\omega}_2 = \alpha\sqrt{\hat{B}_+\hat{B}_- + \beta^2} = \sqrt{\alpha^2\,\hat{H}_1 + (\hbar\Delta)^2}\,. \tag{3.9b}$$

Now, since by initial conditions $\hat{\mathbf{U}}_i(0,0) = \hat{\mathbf{I}}$, then we can write the solution of the evolution matrix differential equation (3.7) as

$$\hat{\mathbf{U}}_i(t,0) = \begin{bmatrix}\cos(\hat{\omega}_1 t) & \sin(\hat{\omega}_1 t)\,\hat{C}\\ \sin(\hat{\omega}_2 t)\,\hat{D} & \cos(\hat{\omega}_2 t)\end{bmatrix}\,, \tag{3.10}$$

and the $\hat{C}$ and $\hat{D}$ operators can be determined by the unitarity conditions

$$\hat{\mathbf{U}}^\dagger_i(t,0)\,\hat{\mathbf{U}}_i(t,0) = \hat{\mathbf{U}}_i(t,0)\,\hat{\mathbf{U}}^\dagger_i(t,0) = \hat{\mathbf{I}}\,. \tag{3.11}$$

In the appendix A we show that the unitarity conditions (3.11) imply

$$\hat{C} = -\hat{D}^\dagger = \frac{i}{\left(\hat{H}_2\right)^{1/4}}\,\sqrt{\hat{T}\hat{B}_-} \tag{3.12a}$$

$$\hat{D} = -\hat{C}^\dagger\,. \tag{3.12b}$$

Therefore, we can write the final expression of the time evolution matrix of the system as

$$\hat{\mathbf{U}}_i(t,0) = \begin{bmatrix}\cos(\hat{\omega}_1 t) & \sin(\hat{\omega}_1 t)\,\hat{C}\\ -\sin(\hat{\omega}_2 t)\,\hat{C}^\dagger & \cos(\hat{\omega}_2 t)\end{bmatrix}\,. \tag{3.13}$$

For Jaynes-Cummings systems an important physical quantity to see how the system under consideration evolves in time is the population inversion factor [11,13,15], defined by

$$\hat{\mathbf{W}}(t) \equiv \hat{\sigma}_+(t)\,\hat{\sigma}_-(t) - \hat{\sigma}_-(t)\,\hat{\sigma}_+(t) = \hat{\sigma}_3(t)\,, \tag{3.14}$$

where the time dependence of the operators is related with the Heisenberg picture. In this case, the time evolution of the population inversion factor will be given by

$$\frac{d\hat{\sigma}_3(t)}{dt} = \frac{1}{i\hbar}\hat{\mathbf{U}}^\dagger_i(t,0)\left[\hat{\sigma}_3, \hat{\mathbf{H}}\right]\hat{\mathbf{U}}_i(t,0)\,, \tag{3.15}$$

and since we have

$$\left[\hat{\sigma}_3, \hat{\mathbf{H}}\right] = \alpha \left[\hat{\sigma}_3, \hat{\mathbf{S}}\right] = -2\alpha \, \hat{\mathbf{S}} \, \hat{\sigma}_3 \,, \tag{3.16}$$

then Eq. (3.15) can be written as

$$\frac{d\hat{\sigma}_3(t)}{dt} = \frac{2i\alpha}{\hbar} \, \hat{\mathbf{S}}(t) \, \hat{\sigma}_3(t) \,. \tag{3.17}$$

We can obtain a differential equation with constant coefficients for $\hat{\sigma}_3(t)$ by taking the time derivative of Eq. (3.17)

$$\frac{d^2\hat{\sigma}_3(t)}{dt^2} = \frac{2i\alpha}{\hbar} \, \left\{ \frac{d\hat{\mathbf{S}}(t)}{dt} \, \hat{\sigma}_3(t) + \hat{\mathbf{S}}(t) \, \frac{d\hat{\sigma}_3(t)}{dt} \right\} \,. \tag{3.18}$$

Having in mind that

$$\frac{d\hat{\mathbf{S}}(t)}{dt} = \frac{1}{i\hbar} \hat{\mathbf{U}}_i^{\dagger}(t,0) \left[\hat{\mathbf{S}}, \hat{\mathbf{H}}\right] \hat{\mathbf{U}}_i(t,0) \,, \tag{3.19}$$

and that

$$\left[\hat{\mathbf{S}}, \hat{\mathbf{H}}\right] = \alpha\beta \left[\hat{\mathbf{S}}, \hat{\sigma}_3\right] = 2\alpha\beta \, \hat{\mathbf{S}} \, \hat{\sigma}_3 \,, \tag{3.20}$$

we conclude that

$$\frac{d\hat{\mathbf{S}}(t)}{dt} = -\frac{2i\alpha\beta}{\hbar} \, \hat{\mathbf{S}}(t) \, \hat{\sigma}_3(t) \,. \tag{3.21}$$

Using Eqs. (3.17) and (3.21) into Eq. (3.18) we obtain

$$\frac{d^2\hat{\sigma}_3(t)}{dt^2} + \hat{\mathbf{\Theta}}^2 \, \hat{\sigma}_3(t) = \hat{\mathbf{F}}(t) \tag{3.22}$$

where

$$\hat{\mathbf{\Theta}}^2 = \frac{4\alpha^2}{\hbar^2} \, \hat{\mathbf{S}}^2 \tag{3.23a}$$

$$\hat{\mathbf{F}}(t) = \frac{4\alpha^2\beta}{\hbar^2} \, \hat{\mathbf{U}}_i^{\dagger}(t,0) \, \hat{\mathbf{S}} \, \hat{\mathbf{U}}_i(t,0) \,. \tag{3.23b}$$

Eq. (3.22) corresponds to a non-homogeneous linear differential equation for $\hat{\sigma}_3(t)$ with constant coefficients since $\hat{\mathbf{S}}^2$ and $\hat{\mathbf{H}}$ commute and, therefore, $\hat{\mathbf{\Theta}}$ is a constant of the motion. The general solution of this differential equation can be written as

$$\hat{\sigma}_3(t) = \hat{\sigma}^H(t) + \hat{\sigma}^P(t) \,, \tag{3.24}$$

and each matrix element of the homogeneous solution, satisfies the differential equation

$$\frac{d^2\hat{\sigma}^H_{jk}(t)}{dt^2} + \hat{\nu}_j^2 \, \hat{\sigma}^H_{jk}(t) = 0 \,, \qquad j,\, k = 1, \text{or } 2 \,, \tag{3.25}$$

with

$$\hbar\hat{\nu}_1 = 2\alpha\sqrt{\hat{T}\hat{B}_-\hat{B}_+\hat{T}^\dagger} = 2\sqrt{\alpha^2\,\hat{H}_2}\,, \tag{3.26a}$$

$$\hbar\hat{\nu}_2 = 2\alpha\sqrt{\hat{B}_+\hat{B}_-} = 2\sqrt{\alpha^2\,\hat{H}_1}\,. \tag{3.26b}$$

The solution of Eq. (3.25) is given by

$$\hat{\sigma}^H_{jk}(t) = \hat{y}_j(t)\;\hat{c}_{jk} + \hat{z}_j(t)\;\hat{d}_{jk}\,, \tag{3.27}$$

where

$$\hat{y}_j(t) = \cos(\hat{\nu}_j t) \tag{3.28a}$$

$$\hat{z}_j(t) = \sin(\hat{\nu}_j t)\,, \tag{3.28b}$$

and the coefficients $\hat{c}_{jk}$ and $\hat{d}_{jk}$ can be determined by the initial conditions.

The matrix elements of the particular solution of the $\hat{\sigma}_3(t)$ differential equation need to satisfy

$$\frac{d^2\hat{\sigma}^P_{jk}(t)}{dt^2} + \hat{\nu}_j^2\,\hat{\sigma}^P_{jk}(t) = \hat{F}_{jk}(t)\,, \qquad j,\,k = 1,\text{or } 2\,, \tag{3.29}$$

and they can be obtained by the variation of parameter or by Green function methods, giving

$$\hat{\sigma}^P_{jk}(t) = \hat{\nu}_j^{-1}\left\{\hat{z}_j(t)\int_0^t d\xi\,\hat{y}_j(\xi)\,\hat{F}_{jk}(\xi) - \hat{y}_j(t)\int_0^t d\xi\,\hat{z}_j(\xi)\,\hat{F}_{jk}(\xi)\right\}\,, \tag{3.30}$$

where we used that the Wronskian of the system of solutions $\hat{y}_j(t)$ and $\hat{z}_j(t)$ is given by $\hat{\nu}_j$.

After we determine the elements of the $\hat{\mathbf{F}}(t)$-matrix, it is necessary to resolve the integrals in Eq. (3.30) to obtain the explicit expression of the particular solution. In the appendix B we show that, using Eqs. (2.2), (3.13), and (3.23), it is possible to conclude that these matrix elements can be written as

$$\begin{aligned}\hat{\sigma}^P_{11}(t) = {}& i\frac{\gamma}{2}\hat{\nu}_1^{-1}\sqrt{\hat{T}\hat{B}_-}\left\{\hat{z}_2(t)\,\mathcal{G}^{(+)}_{CS}(t;\hat{\nu}_2,\hat{\omega}_2,\hat{\omega}_1) - \hat{y}_2(t)\,\mathcal{G}^{(+)}_{SS}(t;\hat{\nu}_2,\hat{\omega}_2,\hat{\omega}_1)\right\}\hat{H}_2^{1/4}\\ &+ i\frac{\gamma}{2}\hat{\nu}_1^{-1}\hat{H}_2^{1/4}\left\{\hat{z}_1(t)\,\mathcal{G}^{(-)}_{SC}(t;\hat{\nu}_1,\hat{\omega}_1,\hat{\omega}_2) - \hat{y}_1(t)\,\mathcal{G}^{(-)}_{CC}(t;\hat{\nu}_1,\hat{\omega}_1,\hat{\omega}_2)\right\}\sqrt{\hat{B}_+\hat{T}^\dagger}\,,\end{aligned} \tag{3.31a}$$

$$\begin{aligned}\hat{\sigma}^P_{12}(t) = {}& \frac{\gamma}{2}\hat{\nu}_1^{-1}\sqrt{\hat{T}\hat{B}_-}\left\{\hat{z}_2(t)\,\mathcal{G}^{(+)}_{CC}(t;\hat{\nu}_2,\hat{\omega}_2,\hat{\omega}_1) - \hat{y}_2(t)\,\mathcal{G}^{(+)}_{SC}(t;\hat{\nu}_2,\hat{\omega}_2,\hat{\omega}_1)\right\}\sqrt{\hat{T}\hat{B}_-}\\ &+ \frac{\gamma}{2}\hat{\nu}_1^{-1}\hat{H}_2^{1/4}\left\{\hat{z}_1(t)\,\mathcal{G}^{(-)}_{SS}(t;\hat{\nu}_1,\hat{\omega}_1,\hat{\omega}_2) + \hat{y}_1(t)\,\mathcal{G}^{(-)}_{CS}(t;\hat{\nu}_1,\hat{\omega}_1,\hat{\omega}_2)\right\}\hat{H}_1^{1/4}\,,\end{aligned} \tag{3.31b}$$

$$\begin{aligned}\hat{\sigma}^P_{21}(t) = {}& \frac{\gamma}{2}\hat{\nu}_2^{-1}\sqrt{\hat{B}_+\hat{T}^\dagger}\left\{\hat{z}_1(t)\,\mathcal{G}^{(+)}_{CC}(t;\hat{\nu}_1,\hat{\omega}_1,\hat{\omega}_2) - \hat{y}_1(t)\,\mathcal{G}^{(+)}_{SC}(t;\hat{\nu}_1,\hat{\omega}_1,\hat{\omega}_2)\right\}\sqrt{\hat{B}_+\hat{T}^\dagger}\\ &+ \frac{\gamma}{2}\hat{\nu}_2^{-1}\hat{H}_1^{1/4}\left\{\hat{z}_2(t)\,\mathcal{G}^{(-)}_{SS}(t;\hat{\nu}_2,\hat{\omega}_2,\hat{\omega}_1) - \hat{y}_2(t)\,\mathcal{G}^{(-)}_{CS}(t;\hat{\nu}_2,\hat{\omega}_2,\hat{\omega}_1)\right\}\hat{H}_2^{1/4}\,,\end{aligned} \tag{3.31c}$$

$$\begin{aligned}\hat{\sigma}^P_{22}(t) = {}& i\frac{\gamma}{2}\hat{\nu}_2^{-1}\sqrt{\hat{B}_+\hat{T}^\dagger}\left\{\hat{z}_1(t)\,\mathcal{G}^{(+)}_{CS}(t;\hat{\nu}_1,\hat{\omega}_1,\hat{\omega}_2) - \hat{y}_1(t)\,\mathcal{G}^{(+)}_{SS}(t;\hat{\nu}_1,\hat{\omega}_1,\hat{\omega}_2)\right\}\hat{H}_1^{1/4}\\ &+ i\frac{\gamma}{2}\hat{\nu}_2^{-1}\hat{H}_1^{1/4}\left\{\hat{z}_2(t)\,\mathcal{G}^{(-)}_{SC}(t;\hat{\nu}_2,\hat{\omega}_2,\hat{\omega}_1) + \hat{y}_2(t)\,\mathcal{G}^{(-)}_{CC}(t;\hat{\nu}_2,\hat{\omega}_2,\hat{\omega}_1)\right\}\sqrt{\hat{T}\hat{B}_-}\,,\end{aligned} \tag{3.31d}$$

where $\gamma = 4\alpha^2\beta/\hbar^2$, and the auxiliary functions are given by

$$\mathcal{G}^{(\pm)}_{XY}(t;\hat{p},\hat{q},\hat{r}) = \mathcal{F}_{XY}(t;\hat{p}-\hat{q},\hat{r}) \pm \mathcal{F}_{XY}(t;\hat{p}+\hat{q},\hat{r})\,, \qquad X,Y = C \text{ or } S\,, \tag{3.32}$$

with

$$\begin{aligned}
\mathcal{F}_{CC}(t;\hat{x},\hat{w}) &\equiv \int_0^t d\xi\, \cos(\hat{x}\xi)\, \cos(\hat{w}\xi) \\
&= \sum_{m,n=0}^{\infty} (-1)^{m+n} \frac{\hat{x}^{2m}\hat{w}^{2n}}{(2m)!\,(2n)!} \frac{t^{2m+2n+1}}{(2m+2n+1)} && (3.33\text{a}) \\
\mathcal{F}_{CS}(t;\hat{x},\hat{w}) &\equiv \int_0^t d\xi\, \cos(\hat{x}\xi)\, \sin(\hat{w}\xi) \\
&= \sum_{m,n=0}^{\infty} (-1)^{m+n} \frac{\hat{x}^{2m}\hat{w}^{2n+1}}{(2m)!\,(2n+1)!} \frac{t^{2m+2n+2}}{(2m+2n+2)} && (3.33\text{b}) \\
\mathcal{F}_{SC}(t;\hat{x},\hat{w}) &\equiv \int_0^t d\xi\, \sin(\hat{x}\xi)\, \cos(\hat{w}\xi) \\
&= \sum_{m,n=0}^{\infty} (-1)^{m+n} \frac{\hat{x}^{2m+1}\hat{w}^{2n}}{(2m+1)!\,(2n)!} \frac{t^{2m+2n+2}}{(2m+2n+2)} && (3.33\text{c}) \\
\mathcal{F}_{SS}(t;\hat{x},\hat{w}) &\equiv \int_0^t d\xi\, \sin(\hat{x}\xi)\, \sin(\hat{w}\xi) \\
&= \sum_{m,n=0}^{\infty} (-1)^{m+n} \frac{\hat{x}^{2m+1}\hat{w}^{2n+1}}{(2m+1)!\,(2n+1)!} \frac{t^{2m+2n+3}}{(2m+2n+3)}\,. && (3.33\text{d})
\end{aligned}$$

With these results for the particular solution we can conclude that

$$\hat{\sigma}^P_{ij}(0) = 0 = \frac{d\hat{\sigma}^P_{ij}(0)}{dt}\,. \tag{3.34}$$

Now, using Eqs. (3.17), (3.24), (3.27), (3.34) and the initial conditions, we have

$$\begin{aligned}
[\hat{\sigma}_3(0)]_{ij} &= \hat{c}_{ij} && (3.35\text{a}) \\
\left[\frac{d\hat{\sigma}_3(0)}{dt}\right]_{ij} &= \frac{2i\alpha}{\hbar}\, \left[\hat{\mathbf{S}}(0)\,\hat{\sigma}_3(0)\right]_{ij} = \hat{\nu}_i\,\hat{d}_{ij}\,. && (3.35\text{b})
\end{aligned}$$

Therefore, the final expression for the elements of the population inversion matrix of the system can be written as

$$[\hat{\sigma}_3(t)]_{ij} = \cos(\hat{\nu}_i t)\; [\hat{\sigma}_3(0)]_{ij} + \frac{2i\alpha}{\hbar}\, \sin(\hat{\nu}_i t)\; \hat{\nu}_i^{-1} \left[\hat{\mathbf{S}}(0)\,\hat{\sigma}_3(0)\right]_{ij} + \hat{\sigma}^P_{ij}(t)\,. \tag{3.36}$$

Again, using these final results we can verify two important and simple limit cases.

a) The Resonant Limit

The first one corresponds to the resonant situation ($\Delta = 0$). Eqs. (3.9), (3.13), (3.26) and (3.31) allow us to conclude that, in this case, the evolution matrix of the system is given by

$$\hat{\mathbf{U}}_i(t,0) = \begin{bmatrix} \cos\left(\frac{1}{2}\hat{\nu}_1 t\right) & \sin\left(\frac{1}{2}\hat{\nu}_1 t\right)\hat{C} \\ -\sin\left(\frac{1}{2}\hat{\nu}_2 t\right)\hat{C}^\dagger & \cos\left(\frac{1}{2}\hat{\nu}_2 t\right) \end{bmatrix}\,. \tag{3.37}$$

and the elements of the population inversion of the system are

$$[\hat{\sigma}_3(t)]_{ij} = \cos(\hat{\nu}_i t)\ [\hat{\sigma}_3(0)]_{ij} + \frac{2i\alpha}{\hbar}\ \sin(\hat{\nu}_i t)\ \hat{\nu}_i^{-1}\left[\hat{\mathbf{S}}(0)\,\hat{\sigma}_3(0)\right]_{ij}\ . \tag{3.38}$$

b) The Standard Jaynes-Cummings Limit

This second important limit corresponds to the case of the harmonic oscillator system, and in this limit we have that $\hat{T} = \hat{T}^\dagger \longrightarrow 1$, $\hat{B}_- \longrightarrow \hat{a}$, $\hat{B}_+ \longrightarrow \hat{a}^\dagger$ and $[\hat{a}, \hat{a}^\dagger] = \hbar\omega$. With these conditions the operators $\hat{\omega}_1$ and $\hat{\omega}_2$ commute, and this fact permits to evaluate the integrals related with the particular solution of the population inversion elements using trigonometric product relations. Using that and the expressions obtained in the appendix B, after a considerable amount of algebra and trigonometric product relations we can show that is possible to write the expressions for the $\hat{\sigma}_{ij}^P(t)$-matrix elements as

$$\begin{aligned}
\hat{\sigma}_{11}^P(t) = {} & i\frac{\gamma}{2}\hat{\nu}_1^{-1}\sqrt{\hat{a}}\left\{\mathcal{K}_S(t;\hat{\omega}_2,\hat{\omega}_1,\hat{\nu}_2) - \mathcal{K}_S(t;\hat{\omega}_2,-\hat{\omega}_1,\hat{\nu}_2)\right\}\left(\hat{a}\hat{a}^\dagger\right)^{1/4} \\
& - i\frac{\gamma}{2}\hat{\nu}_1^{-1}\left(\hat{a}\hat{a}^\dagger\right)^{1/4}\left\{\mathcal{K}_S(t;\hat{\omega}_2,\hat{\omega}_1,\hat{\nu}_1) - \mathcal{K}_S(t;\hat{\omega}_2,-\hat{\omega}_1,\hat{\nu}_1)\right\}\sqrt{\hat{a}^\dagger} && \text{(3.39a)} \\
\hat{\sigma}_{12}^P(t) = {} & \frac{\gamma}{2}\hat{\nu}_1^{-1}\sqrt{\hat{a}}\left\{\mathcal{K}_C(t;\hat{\omega}_2,\hat{\omega}_1,\hat{\nu}_2) - \mathcal{K}_C(t;\hat{\omega}_2,-\hat{\omega}_1,\hat{\nu}_2)\right\}\sqrt{\hat{a}} \\
& - \frac{\gamma}{2}\hat{\nu}_1^{-1}\left(\hat{a}\hat{a}^\dagger\right)^{1/4}\left\{\mathcal{K}_C(t;\hat{\omega}_2,\hat{\omega}_1,\hat{\nu}_1) - \mathcal{K}_C(t;\hat{\omega}_2,-\hat{\omega}_1,\hat{\nu}_1)\right\}\left(\hat{a}^\dagger\hat{a}\right)^{1/4} && \text{(3.39b)} \\
\hat{\sigma}_{21}^P(t) = {} & \frac{\gamma}{2}\hat{\nu}_2^{-1}\sqrt{\hat{a}^\dagger}\left\{\mathcal{K}_C(t;\hat{\omega}_2,\hat{\omega}_1,\hat{\nu}_1) + \mathcal{K}_C(t;\hat{\omega}_2,-\hat{\omega}_1,\hat{\nu}_1)\right\}\sqrt{\hat{a}^\dagger} \\
& - \frac{\gamma}{2}\hat{\nu}_2^{-1}\left(\hat{a}^\dagger\hat{a}\right)^{1/4}\left\{\mathcal{K}_C(t;\hat{\omega}_2,\hat{\omega}_1,\hat{\nu}_2) - \mathcal{K}_C(t;\hat{\omega}_2,-\hat{\omega}_1,\hat{\nu}_2)\right\}\left(\hat{a}\hat{a}^\dagger\right)^{1/4} && \text{(3.39c)} \\
\hat{\sigma}_{22}^P(t) = {} & i\frac{\gamma}{2}\hat{\nu}_2^{-1}\sqrt{\hat{a}^\dagger}\left\{\mathcal{K}_S(t;\hat{\omega}_2,\hat{\omega}_1,\hat{\nu}_1) + \mathcal{K}_S(t;\hat{\omega}_2,-\hat{\omega}_1,\hat{\nu}_1)\right\}\left(\hat{a}^\dagger\hat{a}\right)^{1/4} \\
& - i\frac{\gamma}{2}\hat{\nu}_2^{-1}\left(\hat{a}^\dagger\hat{a}\right)^{1/4}\left\{\mathcal{K}_S(t;\hat{\omega}_2,\hat{\omega}_1,\hat{\nu}_2) + \mathcal{K}_S(t;\hat{\omega}_2,-\hat{\omega}_1,\hat{\nu}_2)\right\}\sqrt{\hat{a}}\,, && \text{(3.39d)}
\end{aligned}$$

where, now, the auxiliary functions are given by

$$\mathcal{K}_S(t;\hat{p},\hat{q},\hat{r}) = \frac{\hat{r}\ \sin\left[(\hat{p}+\hat{q})\,t\right] - (\hat{p}+\hat{q})\ \sin(\hat{r}t)}{\hat{r}^2 - (\hat{p}+\hat{q})^2} \tag{3.40a}$$

$$\mathcal{K}_C(t;\hat{p},\hat{q},\hat{r}) = \frac{\hat{r}\ \cos\left[(\hat{p}+\hat{q})\,t\right] - \hat{r}\ \cos(\hat{r}t)}{\hat{r}^2 - (\hat{p}+\hat{q})^2}\,. \tag{3.40b}$$

Considering the expressions above we may easily verify that the particular solution for the population inversion factor must still satisfy the initial conditions (3.34). Therefore, in this case the final expression for the population inversion factor has the same form given by Eq. (3.36), with

$$\hbar\hat{\nu}_1 = 2\alpha\sqrt{\hat{a}\hat{a}^\dagger}\,, \qquad \hbar\hat{\nu}_2 = 2\alpha\sqrt{\hat{a}^\dagger\hat{a}}\,, \tag{3.41a}$$

$$\hbar\hat{\omega}_1 = \alpha\sqrt{\hat{a}\hat{a}^\dagger + \beta^2}\,, \qquad \hbar\hat{\omega}_2 = \alpha\sqrt{\hat{a}^\dagger\hat{a} + \beta^2}\,. \tag{3.41b}$$

IV. THE GENERALIZED INTENSITY-DEPENDENT NONRESONANT JAYNES-CUMMINGS HAMILTONIAN

A variant of the Jaynes-Cummings model takes the coupling between matter and the radiation to depend on the intensity of the electromagnetic field [13,15,16,18]. This model has great relevance since this kind of interaction means effectively that the coupling is proportional to the amplitude of the field which is a very simple case of a nonlinear interaction corresponding to a more realistic physical situation. The results of this model can also give insight into the behavior of other quantum systems with strong nonlinear interactions. In this section we generalize the standard intensity-dependent nonresonant Jaynes-Cummings model to a shape-invariant one.

The expression for the intensity-dependent nonresonant Jaynes-Cummings Hamiltonian can be written as

$$\hat{\mathbf{H}} = \hat{A}^\dagger \hat{A} + \frac{1}{2}\left[\hat{A}, \hat{A}^\dagger\right](\hat{\sigma}_3 + 1) + \alpha\left(\hat{\sigma}_+ \hat{A}\,\sqrt{\hat{A}^\dagger \hat{A}} + \hat{\sigma}_- \sqrt{\hat{A}^\dagger \hat{A}}\,\hat{A}^\dagger\right) + \hbar\Delta\,\hat{\sigma}_3\,. \tag{4.1}$$

Note that here the constant α is dimensionless. To generalize Hamiltonian (4.1) for all supersymmetric and shape-invariant systems, we can use the operator $\hat{\mathbf{S}}$, given by Eq. (2.2), and further by introducing the following operator

$$\hat{\mathbf{S}}_i = \hat{\sigma}_+ \hat{A}\,\sqrt{\hat{A}^\dagger \hat{A}} + \hat{\sigma}_- \sqrt{\hat{A}^\dagger \hat{A}}\,\hat{A}^\dagger\,. \tag{4.2}$$

Again, the operators $\hat{A}$ and $\hat{A}^\dagger$ satisfy the shape invariance condition, Eq. (1.1). Using operators $\hat{\mathbf{S}}$ and $\hat{\mathbf{S}}_i$ we can decompose the Jaynes-Cummings Hamiltonian in the form

$$\hat{\mathbf{H}} = \hat{\mathbf{H}}_o + \hat{\mathbf{H}}_{int}\,, \tag{4.3}$$

where

$$\hat{\mathbf{H}}_o = \hat{\mathbf{S}}^2\,, \tag{4.4a}$$
$$\hat{\mathbf{H}}_{int} = \alpha\,\hat{\mathbf{S}}_i + \hbar\Delta\,\hat{\sigma}_3\,. \tag{4.4b}$$

In this case, $\hat{\mathbf{H}}_{int}$ can be written as

$$\hat{\mathbf{H}}_{int} = \alpha\begin{bmatrix} \beta & \hat{T}\hat{B}_-\,\sqrt{\hat{B}_+\hat{B}_-} \\ \sqrt{\hat{B}_+\hat{B}_-}\,\hat{B}_+\hat{T}^\dagger & -\beta \end{bmatrix} = \alpha\begin{bmatrix} \beta & \hat{T}\hat{B}_-\,\sqrt{\hat{H}_1} \\ \sqrt{\hat{H}_1}\,\hat{B}_+\hat{T}^\dagger & -\beta \end{bmatrix}. \tag{4.5}$$

Here we can follow the same development of the section II, with the same notation. So, using Eqs. (2.2), (2.9), (4.2) and (4.4), the eigenvalue equation

$$\hat{\mathbf{H}}_{int} \mid \Psi_m^{(\pm)}\rangle = \lambda_m^{(\pm)} \mid \Psi_m^{(\pm)}\rangle\,, \tag{4.6}$$

can be written in a matrix form as

$$\alpha\begin{bmatrix} \beta & \hat{T}\hat{B}_-\,\sqrt{\hat{H}_1} \\ \sqrt{\hat{H}_1}\,\hat{B}_+\hat{T}^\dagger & -\beta \end{bmatrix}\begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix}\begin{bmatrix} C_m^{(\pm)} \mid m\rangle \\ C_{m+1}^{(\pm)} \mid m+1\rangle \end{bmatrix} = \lambda_m^{(\pm)}\begin{bmatrix} C_m^{(\pm)} \mid m\rangle \\ C_{m+1}^{(\pm)} \mid m+1\rangle \end{bmatrix}. \tag{4.7}$$

Again, since the C's coefficients commute with the $\hat{A}$ or $\hat{A}^\dagger$ operators, then the last matrix equation permits to obtain the following equations

$$\left[\alpha\beta - \lambda_m^{(\pm)}\right]\left(\hat{T}C_m^{(\pm)}\hat{T}^\dagger\right)\hat{T} \mid m\rangle \pm \alpha\, C_{m+1}^{(\pm)}\hat{T}\hat{B}_- \sqrt{\hat{H}_1} \mid m+1\rangle = 0 \tag{4.8a}$$

$$\alpha\left(\hat{T}C_m^{(\pm)}\hat{T}^\dagger\right)\sqrt{\hat{H}_1}\;\hat{B}_+ \mid m\rangle \mp \left[\alpha\beta + \lambda_m^{(\pm)}\right] C_{m+1}^{(\pm)} \mid m+1\rangle = 0\,. \tag{4.8b}$$

Now, using from Eqs. (2.15) to (2.17) we have

$$\begin{aligned}\hat{T}\hat{B}_- \sqrt{\hat{H}_1} \mid m+1\rangle &= \hat{T}\hat{B}_- \sqrt{\mathcal{E}_{m+1}} \mid m+1\rangle \\ &= \sqrt{\mathcal{E}_{m+1}}\;\hat{T}\hat{B}_- \mid m+1\rangle \\ &= \mathcal{E}_{m+1}\;\hat{T} \mid m\rangle\,,\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}\sqrt{\hat{H}_1}\;\hat{B}_+ \mid m\rangle &= \sqrt{\hat{H}_1}\sqrt{\mathcal{E}_{m+1}} \mid m+1\rangle \\ &= \sqrt{\mathcal{E}_{m+1}}\sqrt{\hat{H}_1} \mid m+1\rangle \\ &= \mathcal{E}_{m+1} \mid m+1\rangle\,,\end{aligned} \tag{4.10}$$

Using Eqs. (4.9) and (4.10), then Eqs. (4.8) take the form

$$\left\{\left[\alpha\beta - \lambda_m^{(\pm)}\right]\left(\hat{T}C_m^{(\pm)}\hat{T}^\dagger\right) \pm \alpha\;\mathcal{E}_{m+1}\,C_{m+1}^{(\pm)}\right\}\hat{T} \mid m\rangle = 0 \tag{4.11a}$$

$$\left\{\alpha\;\mathcal{E}_{m+1}\left(\hat{T}C_m^{(\pm)}\hat{T}^\dagger\right) \mp \left[\alpha\beta + \lambda_m^{(\pm)}\right] C_{m+1}^{(\pm)}\right\} \mid m+1\rangle = 0\,. \tag{4.11b}$$

From Eqs. (4.11) it follows that

$$\lambda_m^{(\pm)} = \pm\alpha\sqrt{\mathcal{E}_{m+1}^2 + \beta^2}\,, \tag{4.12}$$

and

$$C_{m+1}^{(\pm)} = \left(\frac{\sqrt{\mathcal{E}_{m+1}^2 + \beta^2} \mp \beta}{\mathcal{E}_{m+1}}\right)\left(\hat{T}C_m^{(\pm)}\hat{T}^\dagger\right)\,. \tag{4.13}$$

Eqs. (2.11) and (4.13) imply that

$$C_{m+1}^{(\pm)} = C_m^{(\mp)}\,, \tag{4.14}$$

and the eigenstates and eigenvalues of the generalized intensity-dependent nonresonant Jaynes-Cummings Hamiltonian can be written as

$$E_m^{(\pm)} = \mathcal{E}_{m+1} \pm \sqrt{\alpha^2\;\mathcal{E}_{m+1}^2 + \hbar^2\Delta^2}\,, \tag{4.15}$$

and

$$| \Psi_m^{(\pm)} \rangle = \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} C_m^{(\pm)} \mid m \rangle \\ C_m^{(\mp)} \mid m+1 \rangle \end{bmatrix}, \qquad m = 0, 1, 2, \cdots \tag{4.16}$$

a) The Intensity-Dependent Resonant Limit

From these general results we can again verify the two simple limiting cases. The first one, corresponding to the resonant situation, is for $\Delta = 0$ $(\beta = 0)$. Using these conditions into Eqs. (4.13) and (4.15) and Eqs. (2.11) we can promptly conclude that

$$E_m^{(\pm)} = (1 \pm \alpha)\, \mathcal{E}_{m+1}\,, \tag{4.17}$$

and

$$C_{m+1}^{(\pm)} = \hat{T} C_m^{(\pm)} \hat{T}^\dagger = C_m^{(\pm)} = \frac{1}{\sqrt{2}}\,. \tag{4.18}$$

Therefore the intensity-dependent resonant Jaynes-Cummings eigenstate is given by

$$| \Psi_m^{(\pm)} \rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} \mid m \rangle \\ \mid m+1 \rangle \end{bmatrix}, \qquad m = 0, 1, 2, \cdots\cdots\,. \tag{4.19}$$

If we compare this last particular result with that one found in the reference [8], we conclude that the intensity-dependent and intensity-independent generalized Jaynes-Cummings Hamiltonians have the same eigenstates in the resonant situation.

b) The Standard Intensity-Dependent Jaynes-Cummings Limit

The second limit, corresponding to the standard intensity-dependent Jaynes-Cummings case, is related with the harmonic oscillator system. In this limit we have that $\hat{T} = \hat{T}^\dagger \longrightarrow 1$, $\hat{B}_- \longrightarrow \hat{a}$, $\hat{B}_+ \longrightarrow \hat{a}^\dagger$, $\Delta = \omega - \omega_o$ and $\mathcal{E}_{m+1} = (m+1)\hbar\omega$. Using these conditions in Eqs. (4.13), (4.15) and Eqs. (2.11) we can promptly conclude that

$$E_m^{(\pm)} = (m+1)\hbar\omega \pm \hbar\sqrt{\alpha^2\omega^2(m+1)^2 + (\omega - \omega_o)^2}\,, \tag{4.20}$$

and

$$C_{m+1}^{(\pm)} = \gamma_m^{(\pm)} C_m^{(\pm)} = C_m^{(\mp)} = \frac{1}{\sqrt{1 + \left(\gamma_m^{(\mp)}\right)^2}}\,, \tag{4.21}$$

where

$$\gamma_m^{(\pm)} = \sqrt{1 + \delta_m^2} \mp \delta_m\,, \tag{4.22a}$$

$$\delta_m = \frac{\omega - \omega_o}{\alpha\omega(m+1)}\,. \tag{4.22b}$$

Therefore the standard intensity-dependent Jaynes-Cummings eigenstate, written in a matrix form, is given by

$$| \Psi_m^{(\pm)} \rangle = \frac{1}{\sqrt{1 + \left(\gamma_m^{(\pm)}\right)^2}} \begin{bmatrix} 1 & 0 \\ 0 & \pm\gamma_m^{(\pm)} \end{bmatrix} \begin{bmatrix} \mid m \rangle \\ \mid m+1 \rangle \end{bmatrix}, \qquad m = 0, 1, 2, \cdots\,. \tag{4.23}$$

V. TIME EVOLUTION OF THE INTENSITY-DEPENDENT NONRESONANT SYSTEM

To resolve the time-dependent Schrödinger equation for intensity-dependent nonresonant Jaynes-Cummings systems:

$$i\hbar \frac{\partial}{\partial t} \mid \Psi(t)\rangle = \left(\hat{\mathbf{H}}_o + \hat{\mathbf{H}}_{int}\right) \mid \Psi(t)\rangle \tag{5.1}$$

we can again write the state $\mid \Psi(t)\rangle$ as it is given by Eq. (3.2). Then using from Eqs. (3.3) to (3.5), we can write the matrix equation

$$i\hbar \begin{bmatrix} \hat{U}'_{11} & \hat{U}'_{12} \\ \hat{U}'_{21} & \hat{U}'_{22} \end{bmatrix} = \alpha \begin{bmatrix} \beta & \hat{T}\hat{B}_- \sqrt{\hat{H}_1} \\ \sqrt{\hat{H}_1}\, \hat{B}_+\hat{T}^\dagger & -\beta \end{bmatrix} \begin{bmatrix} \hat{U}_{11} & \hat{U}_{12} \\ \hat{U}_{21} & \hat{U}_{22} \end{bmatrix}, \tag{5.2}$$

To diagonalize this evolution matrix differential equation we can differentiate Eq. (3.5) with respect to time. After that, if we again use the same Eq. (3.5), we find

$$i\hbar \frac{\partial^2}{\partial t^2} \hat{\mathbf{U}}_i(t,0) = \hat{\mathbf{H}}_{int} \frac{\partial}{\partial t} \hat{\mathbf{U}}_i(t,0) = \frac{1}{i\hbar} \hat{\mathbf{H}}^2_{int} \hat{\mathbf{U}}_i(t,0)\,, \tag{5.3}$$

which can be written as

$$\begin{bmatrix} \hat{U}''_{11} & \hat{U}''_{12} \\ \hat{U}''_{21} & \hat{U}''_{22} \end{bmatrix} = - \begin{bmatrix} \hat{\omega}_1 & 0 \\ 0 & \hat{\omega}_2 \end{bmatrix} \begin{bmatrix} \hat{U}_{11} & \hat{U}_{12} \\ \hat{U}_{21} & \hat{U}_{22} \end{bmatrix}, \tag{5.4}$$

where

$$\hbar\hat{\omega}_1 = \alpha \sqrt{(\hat{T}\hat{B}_-\hat{B}_+\hat{T}^\dagger)^2 + \beta^2} = \sqrt{\alpha^2\, \hat{H}_2^2 + (\hbar\Delta)^2}\,, \tag{5.5a}$$

$$\hbar\hat{\omega}_2 = \alpha \sqrt{(\hat{B}_+\hat{B}_-)^2 + \beta^2} = \sqrt{\alpha^2\, \hat{H}_1^2 + (\hbar\Delta)^2}\,. \tag{5.5b}$$

Now, using the initial conditions $\hat{\mathbf{U}}_i(0,0) = \hat{\mathbf{I}}$, we can write the solution of the evolution matrix differential equation (5.3) as

$$\hat{\mathbf{U}}_i(t,0) = \begin{bmatrix} \cos(\hat{\omega}_1 t) & \sin(\hat{\omega}_1 t)\, \hat{C} \\ \sin(\hat{\omega}_2 t)\, \hat{D} & \cos(\hat{\omega}_2 t) \end{bmatrix}, \tag{5.6}$$

where the $\hat{C}$ and $\hat{D}$ operators can be determined by Eq. (3.11). Following the same steps used in the appendix A, we can conclude that these operators must have the form given by Eqs. (3.12). So, in this case the final expression of the time evolution matrix $\hat{\mathbf{U}}_i(t,0)$ is given by Eq. (3.13) as well.

To obtain the population inversion factor we can again follow the steps from Eq. (3.14) to (3.30), but replacing the operator $\hat{\mathbf{S}}$ by the operator $\hat{\mathbf{S}}_i$. Besides that we have

$$\hbar\hat{\nu}_1 = 2\alpha\, \hat{T}\hat{B}_-\hat{B}_+\hat{T}^\dagger = 2\alpha\, \hat{H}_2\,, \tag{5.7a}$$

$$\hbar\hat{\nu}_2 = 2\alpha\, \hat{B}_+\hat{B}_- = 2\alpha\, \hat{H}_1\,, \tag{5.7b}$$

instead Eqs. (3.26). Here, we can again use the development shown in the appendix B, just replacing $\hat{\mathbf{S}}$ by $\hat{\mathbf{S}}_i$, to obtain the explicit form of the matrix elements for the particular solution of the population inversion factor, given by Eq. (3.30). So for a shape-invariant intensity-dependent nonresonant Jaynes-Cummings system, these matrix elements are given by

$$\hat{\sigma}^P_{11}(t) = i\frac{\gamma}{2}\hat{\nu}_1^{-1}\sqrt{\hat{T}\hat{B}_-}\left\{\hat{z}_2(t)\,\mathcal{G}^{(+)}_{CS}(t;\hat{\nu}_2,\hat{\omega}_2,\hat{\omega}_1) - \hat{y}_2(t)\,\mathcal{G}^{(+)}_{SS}(t;\hat{\nu}_2,\hat{\omega}_2,\hat{\omega}_1)\right\}\hat{H}_2^{3/4} + i\frac{\gamma}{2}\hat{\nu}_1^{-1}\hat{H}_2^{3/4}\left\{\hat{z}_1(t)\,\mathcal{G}^{(-)}_{SC}(t;\hat{\nu}_1,\hat{\omega}_1,\hat{\omega}_2) - \hat{y}_1(t)\,\mathcal{G}^{(-)}_{CC}(t;\hat{\nu}_1,\hat{\omega}_1,\hat{\omega}_2)\right\}\sqrt{\hat{B}_+\hat{T}^\dagger}\,, \tag{5.8a}$$

$$\hat{\sigma}^P_{12}(t) = \frac{\gamma}{2}\hat{\nu}_1^{-1}\sqrt{\hat{T}\hat{B}_-}\left\{\hat{z}_2(t)\,\mathcal{G}^{(+)}_{CC}(t;\hat{\nu}_2,\hat{\omega}_2,\hat{\omega}_1) - \hat{y}_2(t)\,\mathcal{G}^{(+)}_{SC}(t;\hat{\nu}_2,\hat{\omega}_2,\hat{\omega}_1)\right\}\sqrt{\hat{H}_2\hat{T}\hat{B}_-} + \frac{\gamma}{2}\hat{\nu}_1^{-1}\hat{H}_2^{3/4}\left\{\hat{z}_1(t)\,\mathcal{G}^{(-)}_{SS}(t;\hat{\nu}_1,\hat{\omega}_1,\hat{\omega}_2) + \hat{y}_1(t)\,\mathcal{G}^{(-)}_{CS}(t;\hat{\nu}_1,\hat{\omega}_1,\hat{\omega}_2)\right\}\hat{H}_1^{1/4}\,, \tag{5.8b}$$

$$\hat{\sigma}^P_{21}(t) = \frac{\gamma}{2}\hat{\nu}_2^{-1}\sqrt{\hat{B}_+\hat{T}^\dagger\hat{H}_2}\left\{\hat{z}_1(t)\,\mathcal{G}^{(+)}_{CC}(t;\hat{\nu}_1,\hat{\omega}_1,\hat{\omega}_2) - \hat{y}_1(t)\,\mathcal{G}^{(+)}_{SC}(t;\hat{\nu}_1,\hat{\omega}_1,\hat{\omega}_2)\right\}\sqrt{\hat{B}_+\hat{T}^\dagger} + \frac{\gamma}{2}\hat{\nu}_2^{-1}\hat{H}_1^{1/4}\left\{\hat{z}_2(t)\,\mathcal{G}^{(-)}_{SS}(t;\hat{\nu}_2,\hat{\omega}_2,\hat{\omega}_1) - \hat{y}_2(t)\,\mathcal{G}^{(-)}_{CS}(t;\hat{\nu}_2,\hat{\omega}_2,\hat{\omega}_1)\right\}\hat{H}_2^{3/4}\,, \tag{5.8c}$$

$$\hat{\sigma}^P_{22}(t) = i\frac{\gamma}{2}\hat{\nu}_2^{-1}\sqrt{\hat{B}_+\hat{T}^\dagger\hat{H}_2}\left\{\hat{z}_1(t)\,\mathcal{G}^{(+)}_{CS}(t;\hat{\nu}_1,\hat{\omega}_1,\hat{\omega}_2) - \hat{y}_1(t)\,\mathcal{G}^{(+)}_{SS}(t;\hat{\nu}_1,\hat{\omega}_1,\hat{\omega}_2)\right\}\hat{H}_1^{1/4} + i\frac{\gamma}{2}\hat{\nu}_2^{-1}\hat{H}_1^{1/4}\left\{\hat{z}_2(t)\,\mathcal{G}^{(-)}_{SC}(t;\hat{\nu}_2,\hat{\omega}_2,\hat{\omega}_1) + \hat{y}_2(t)\,\mathcal{G}^{(-)}_{CC}(t;\hat{\nu}_2,\hat{\omega}_2,\hat{\omega}_1)\right\}\sqrt{\hat{H}_2\hat{T}\hat{B}_-}\,. \tag{5.8d}$$

Yet the auxiliary functions, $\mathcal{G}^{(\pm)}_{XY}(t;\hat{p},\hat{q},\hat{r})$, are given by Eqs. (3.32) and (3.33). From Eqs. (3.34) and (3.35), we have for the elements of the population inversion matrix:

$$[\hat{\sigma}_3(t)]_{ij} = \cos(\hat{\nu}_i t)\ [\hat{\sigma}_3(0)]_{ij} + \frac{2i\alpha}{\hbar}\sin(\hat{\nu}_i t)\ \hat{\nu}_i^{-1}\left[\hat{\mathbf{S}}_i(0)\,\hat{\sigma}_3(0)\right]_{ij} + \hat{\sigma}^P_{ij}(t)\,. \tag{5.9}$$

a) The Intensity-Dependent Resonant Limit

In this limit we set ($\Delta = 0$), so the evolution matrix of the system is given by

$$\hat{\mathbf{U}}_i(t,0) = \begin{bmatrix} \cos\left(\frac{1}{2}\hat{\nu}_1 t\right) & \sin\left(\frac{1}{2}\hat{\nu}_1 t\right)\hat{C} \\ -\sin\left(\frac{1}{2}\hat{\nu}_2 t\right)\hat{C}^\dagger & \cos\left(\frac{1}{2}\hat{\nu}_2 t\right) \end{bmatrix}. \tag{5.10}$$

and the elements of the population inversion factor can be written as

$$[\hat{\sigma}_3(t)]_{ij} = \cos(\hat{\nu}_i t)\ [\hat{\sigma}_3(0)]_{ij} + \frac{2i\alpha}{\hbar}\sin(\hat{\nu}_i t)\ \hat{\nu}_i^{-1}\left[\hat{\mathbf{S}}_i(0)\,\hat{\sigma}_3(0)\right]_{ij}\,. \tag{5.11}$$

b) The Standard Intensity-Dependent Jaynes-Cummings Limit

For the case of a harmonic oscillator system ($\hat{T} = \hat{T}^\dagger \longrightarrow 1$, $\hat{B}_- \longrightarrow \hat{a}$, $\hat{B}_+ \longrightarrow \hat{a}^\dagger$ and $[\hat{a},\hat{a}^\dagger] = \hbar\omega$), we have for the $\hat{\sigma}^P_{ij}(t)$-matrix elements the following expressions

$$\hat{\sigma}^P_{11}(t) = i\frac{\gamma}{2}\hat{\nu}_1^{-1}\sqrt{\hat{a}}\left\{\mathcal{K}_S(t;\hat{\omega}_2,\hat{\omega}_1,\hat{\nu}_2) - \mathcal{K}_S(t;\hat{\omega}_2,-\hat{\omega}_1,\hat{\nu}_2)\right\}\left(\hat{a}\hat{a}^\dagger\right)^{3/4} - i\frac{\gamma}{2}\hat{\nu}_1^{-1}\left(\hat{a}\hat{a}^\dagger\right)^{3/4}\left\{\mathcal{K}_S(t;\hat{\omega}_2,\hat{\omega}_1,\hat{\nu}_1) - \mathcal{K}_S(t;\hat{\omega}_2,-\hat{\omega}_1,\hat{\nu}_1)\right\}\sqrt{\hat{a}^\dagger} \tag{5.12a}$$

$$
\begin{aligned}
\hat{\sigma}^P_{12}(t) &= \frac{\gamma}{2}\hat{\nu}_1^{-1}\sqrt{\hat{a}}\left\{\mathcal{K}_C(t;\hat{\omega}_2,\hat{\omega}_1,\hat{\nu}_2) - \mathcal{K}_C(t;\hat{\omega}_2,-\hat{\omega}_1,\hat{\nu}_2)\right\}\sqrt{\hat{a}\hat{a}^\dagger\hat{a}} \\
&\quad - \frac{\gamma}{2}\hat{\nu}_1^{-1}\left(\hat{a}\hat{a}^\dagger\right)^{3/4}\left\{\mathcal{K}_C(t;\hat{\omega}_2,\hat{\omega}_1,\hat{\nu}_1) - \mathcal{K}_C(t;\hat{\omega}_2,-\hat{\omega}_1,\hat{\nu}_1)\right\}\left(\hat{a}^\dagger\hat{a}\right)^{1/4} \qquad (5.12b) \\
\hat{\sigma}^P_{21}(t) &= \frac{\gamma}{2}\hat{\nu}_2^{-1}\sqrt{\hat{a}^\dagger\hat{a}\hat{a}^\dagger}\left\{\mathcal{K}_C(t;\hat{\omega}_2,\hat{\omega}_1,\hat{\nu}_1) + \mathcal{K}_C(t;\hat{\omega}_2,-\hat{\omega}_1,\hat{\nu}_1)\right\}\sqrt{\hat{a}^\dagger} \\
&\quad - \frac{\gamma}{2}\hat{\nu}_2^{-1}\left(\hat{a}^\dagger\hat{a}\right)^{1/4}\left\{\mathcal{K}_C(t;\hat{\omega}_2,\hat{\omega}_1,\hat{\nu}_2) - \mathcal{K}_C(t;\hat{\omega}_2,-\hat{\omega}_1,\hat{\nu}_2)\right\}\left(\hat{a}\hat{a}^\dagger\right)^{3/4} \qquad (5.12c) \\
\hat{\sigma}^P_{22}(t) &= i\frac{\gamma}{2}\hat{\nu}_2^{-1}\sqrt{\hat{a}^\dagger\hat{a}\hat{a}^\dagger}\left\{\mathcal{K}_S(t;\hat{\omega}_2,\hat{\omega}_1,\hat{\nu}_1) + \mathcal{K}_S(t;\hat{\omega}_2,-\hat{\omega}_1,\hat{\nu}_1)\right\}\left(\hat{a}^\dagger\hat{a}\right)^{1/4} \\
&\quad - i\frac{\gamma}{2}\hat{\nu}_2^{-1}\left(\hat{a}^\dagger\hat{a}\right)^{1/4}\left\{\mathcal{K}_S(t;\hat{\omega}_2,\hat{\omega}_1,\hat{\nu}_2) + \mathcal{K}_S(t;\hat{\omega}_2,-\hat{\omega}_1,\hat{\nu}_2)\right\}\sqrt{\hat{a}\hat{a}^\dagger\hat{a}}\,, \qquad (5.12d)
\end{aligned}
$$

where the auxiliary functions, $\mathcal{K}_S(t;\hat{p},\hat{q},\hat{r})$ and $\mathcal{K}_C(t;\hat{p},\hat{q},\hat{r})$, are given by Eqs. (3.40). The final expression for the population inversion factor has the same form given by Eq. (5.9) with

$$
\hbar\hat{\nu}_1 = 2\alpha\,\hat{a}\hat{a}^\dagger\,, \qquad \hbar\hat{\nu}_2 = 2\alpha\,\hat{a}^\dagger\hat{a}\,, \qquad (5.13a)
$$

$$
\hbar\hat{\omega}_1 = \alpha\sqrt{(\hat{a}\hat{a}^\dagger)^2+\beta^2}\,, \qquad \hbar\hat{\omega}_2 = \alpha\sqrt{(\hat{a}^\dagger\hat{a})^2+\beta^2}\,. \qquad (5.13b)
$$

VI. CONCLUSIONS

In this article we extended our earlier work [8] on bound-state problems which represent two-level systems. The corresponding coupled-channel Hamiltonians generalize the nonresonant and intensity-dependent nonresonant Jaynes-Cummings Hamiltonians. In the case of a nonresonant system, if we take the starting Hamiltonian to be the simplest shape-invariant system, namely the harmonic oscillator, our results reduce to those of the standard nonresonant Jaynes-Cummings approach, which has been extensively used to model a two-level atom-single field mode interaction whose detuning it is not null. In addition we have studied time evolution and population inversion factor of the both kind of generalized systems.

These models are not only interesting on their own account. Being exactly solvable coupled-channels problems they may help to assess the validity and accuracy of various approximate approaches to the coupled-channel problems [19].

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Appendix A

Here we give the steps used to obtain the specific form of the operators $\hat{C}$ and $\hat{D}$. Using Eq. (3.10) into the unitary condition equation (3.11) actually we can show that the $\hat{C}$ and $\hat{D}$ operators need to satisfy the following six conditions

$$\hat{C}\hat{C}^\dagger = \hat{C}^\dagger\hat{C} = 1 \tag{1a}$$
$$\hat{D}\hat{D}^\dagger = \hat{D}^\dagger\hat{D} = 1 \tag{1b}$$
$$\hat{D}^\dagger \sin(\hat{\omega}_2 t) = -\sin(\hat{\omega}_1 t)\ \hat{C} \tag{1c}$$
$$\hat{D} \cos(\hat{\omega}_1 t) = -\cos(\hat{\omega}_2 t)\ \hat{C}^\dagger\,. \tag{1d}$$

At this point we can use the following property

$$\begin{aligned}
\sqrt{\hat{T}\hat{B}_-}\ \hat{\omega}_2 &= \sqrt{\hat{T}\hat{B}_-}\ \sqrt{\alpha^2\hat{B}_+\hat{B}_- + \beta^2}/\hbar \\
&= \sqrt{\alpha^2\hat{T}\hat{B}_-\hat{B}_+\hat{B}_- + \hat{T}\hat{B}_-\beta^2}/\hbar \\
&= \sqrt{\alpha^2\hat{T}\hat{B}_-\hat{B}_+\hat{T}^\dagger\hat{T}\hat{B}_- + \beta^2\hat{T}\hat{B}_-}/\hbar \\
&= \sqrt{\alpha^2\hat{T}\hat{B}_-\hat{B}_+\hat{T}^\dagger + \beta^2}/\hbar\ \sqrt{\hat{T}\hat{B}_-} \\
&= \hat{\omega}_1\ \sqrt{\hat{T}\hat{B}_-}\,.
\end{aligned} \tag{2}$$

Then, with this result we have

$$\sqrt{\hat{T}\hat{B}_-}\ \hat{\omega}_2^2 = \sqrt{\hat{T}\hat{B}_-}\ \hat{\omega}_2\ \hat{\omega}_2 = \hat{\omega}_1\ \sqrt{\hat{T}\hat{B}_-}\ \hat{\omega}_2 = \hat{\omega}_1^2\ \sqrt{\hat{T}\hat{B}_-}\,, \tag{3}$$

and finally, by induction, we conclude that

$$\sqrt{\hat{T}\hat{B}_-}\ \hat{\omega}_2^n = \hat{\omega}_1^n\ \sqrt{\hat{T}\hat{B}_-}\,. \tag{4}$$

In the same way,

$$\begin{aligned}
\sqrt{\hat{B}_+\hat{T}^\dagger}\ \hat{\omega}_1 &= \sqrt{\hat{B}_+\hat{T}^\dagger}\ \sqrt{\alpha^2\hat{T}\hat{B}_-\hat{B}_+\hat{T}^\dagger + \beta^2}/\hbar \\
&= \sqrt{\alpha^2\hat{B}_+\hat{T}^\dagger\hat{T}\hat{B}_-\hat{B}_+\hat{T}^\dagger + \hat{B}_+\hat{T}^\dagger\beta^2}/\hbar \\
&= \sqrt{\alpha^2\hat{B}_+\hat{B}_-\hat{B}_+\hat{T}^\dagger + \beta^2\hat{B}_+\hat{T}^\dagger}/\hbar \\
&= \sqrt{\alpha^2\hat{B}_+\hat{B}_- + \beta^2}/\hbar\ \sqrt{\hat{B}_+\hat{T}^\dagger} \\
&= \hat{\omega}_2\ \sqrt{\hat{B}_+\hat{T}^\dagger}\,.
\end{aligned} \tag{5}$$

Then, with this result we have

$$\sqrt{\hat{B}_+\hat{T}^\dagger}\ \hat{\omega}_1^2 = \sqrt{\hat{B}_+\hat{T}^\dagger}\ \hat{\omega}_1\ \hat{\omega}_1 = \hat{\omega}_2\ \sqrt{\hat{B}_+\hat{T}^\dagger}\ \hat{\omega}_1 = \hat{\omega}_2^2\ \sqrt{\hat{B}_+\hat{T}^\dagger}\,, \tag{6}$$

and finally, again by induction, we get

$$\sqrt{\hat{B}_+\hat{T}^\dagger}\ \hat{\omega}_1^n = \hat{\omega}_2^n\ \sqrt{\hat{B}_+\hat{T}^\dagger}\,. \tag{7}$$

Using the properties given by Eqs. (4) and (7) and the forms of $\hat{C}$, $\hat{D}$ operators, defined by Eqs. (3.12), we can verify that

$$\hat{C}\hat{C}^\dagger = \hat{D}^\dagger\hat{D} = \frac{i}{\hat{H}_2^{1/4}}\sqrt{\hat{T}\hat{B}_-}\,\sqrt{\hat{B}_+\hat{T}^\dagger}\frac{(-i)}{\hat{H}_2^{1/4}} = \frac{1}{\hat{H}_2^{1/4}}\sqrt{\hat{H}_2}\frac{1}{\hat{H}_2^{1/4}} = 1\,, \tag{8}$$

and

$$\hat{C}^\dagger\hat{C} = \hat{D}\hat{D}^\dagger = \sqrt{\hat{B}_+\hat{T}^\dagger}\frac{(-i)}{\hat{H}_2^{1/4}}\,\frac{i}{\hat{H}_2^{1/4}}\sqrt{\hat{T}\hat{B}_-} = \sqrt{\hat{B}_+\hat{T}^\dagger}\frac{1}{\sqrt{\hat{H}_2}}\,\sqrt{\hat{H}_2}\frac{1}{\sqrt{\hat{B}_+\hat{T}^\dagger}} = 1\,. \tag{9}$$

Also using the series expansion of the trigonometric functions, we can show that

$$\begin{aligned}
\hat{D}^\dagger\,\sin(\hat{\omega}_2 t) &= \frac{-i}{\hat{H}_2^{1/4}}\sqrt{\hat{T}\hat{B}_-}\sum_{n=0}^{\infty}(-1)^n\frac{(\hat{\omega}_2 t)^{2n+1}}{(2n+1)!}\\
&= \frac{-i}{\hat{H}_2^{1/4}}\sum_{n=0}^{\infty}(-1)^n\sqrt{\hat{T}\hat{B}_-}\frac{(\hat{\omega}_2 t)^{2n+1}}{(2n+1)!}\\
&= \frac{-i}{\hat{H}_2^{1/4}}\sum_{n=0}^{\infty}(-1)^n\frac{(\hat{\omega}_1 t)^{2n+1}}{(2n+1)!}\sqrt{\hat{T}\hat{B}_-}\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{(\hat{\omega}_1 t)^{2n+1}}{(2n+1)!}\frac{-i}{\hat{H}_2^{1/4}}\sqrt{\hat{T}\hat{B}_-}\\
&= -\sin(\hat{\omega}_1 t)\,\hat{C}\,,
\end{aligned} \tag{10}$$

where we used the commutation between $\hat{H}_2$ and $\hat{\omega}_1$ (see Appendix B). In the same way we can prove that

$$\begin{aligned}
\hat{D}\,\cos(\hat{\omega}_1 t) &= \sqrt{\hat{B}_+\hat{T}^\dagger}\frac{i}{\hat{H}_2^{1/4}}\sum_{n=0}^{\infty}(-1)^n\frac{(\hat{\omega}_1 t)^{2n}}{(2n)!}\\
&= \sum_{n=0}^{\infty}(-1)^n\sqrt{\hat{B}_+\hat{T}^\dagger}\frac{(\hat{\omega}_1 t)^{2n}}{(2n)!}\frac{i}{\hat{H}_2^{1/4}}\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{(\hat{\omega}_2 t)^{2n}}{(2n)!}\sqrt{\hat{B}_+\hat{T}^\dagger}\frac{i}{\hat{H}_2^{1/4}}\\
&= -\cos(\hat{\omega}_2 t)\,\hat{C}^\dagger\,.
\end{aligned} \tag{11}$$

Again, we used the commutation between $\hat{H}_2$ and $\hat{\omega}_1$.

Appendix B

In this appendix we show the necessary steps to obtain the explicit expressions of the particular solution elements of the population inversion factor. To resolve the integrals in Eq. (3.30), first we need to determine the elements of the $\hat{\mathbf{F}}(t)$-matrix. To do that we can use Eqs. (2.2), (3.23), and (3.13) to write down

$$\hat{F}_{11}(t) = -\gamma \left\{ \cos(\hat{\omega}_1 t)\, \hat{T}\hat{B}_- \, \sin(\hat{\omega}_2 t)\, \hat{C}^\dagger + \hat{C}\, \sin(\hat{\omega}_2 t)\, \hat{B}_+\hat{T}^\dagger \, \cos(\hat{\omega}_1 t) \right\}$$
$$= i\gamma \left\{ \sqrt{\hat{T}\hat{B}_-}\, \cos(\hat{\omega}_2 t)\, \sin(\hat{\omega}_1 t)\, \hat{H}_2^{1/4} - \hat{H}_2^{1/4}\, \sin(\hat{\omega}_1 t)\, \cos(\hat{\omega}_2 t)\, \sqrt{\hat{B}_+\hat{T}^\dagger} \right\} \tag{12a}$$
$$\hat{F}_{12}(t) = \gamma \left\{ \cos(\hat{\omega}_1 t)\, \hat{T}\hat{B}_- \, \cos(\hat{\omega}_2 t) - \hat{C}\, \sin(\hat{\omega}_2 t)\, \hat{B}_+\hat{T}^\dagger \, \sin(\hat{\omega}_1 t)\, \hat{C} \right\}$$
$$= \gamma \left\{ \sqrt{\hat{T}\hat{B}_-}\, \cos(\hat{\omega}_2 t)\, \cos(\hat{\omega}_1 t)\, \sqrt{\hat{T}\hat{B}_-} + \hat{H}_2^{1/4}\, \sin(\hat{\omega}_1 t)\, \sin(\hat{\omega}_2 t)\, \hat{H}_1^{1/4} \right\} \tag{12b}$$
$$\hat{F}_{21}(t) = \gamma \left\{ \cos(\hat{\omega}_2 t)\, \hat{B}_+\hat{T}^\dagger \, \cos(\hat{\omega}_1 t) - \hat{C}^\dagger\, \sin(\hat{\omega}_1 t)\, \hat{T}\hat{B}_- \, \sin(\hat{\omega}_2 t)\, \hat{C}^\dagger \right\}$$
$$= \gamma \left\{ \sqrt{\hat{B}_+\hat{T}^\dagger}\, \cos(\hat{\omega}_1 t)\, \cos(\hat{\omega}_2 t)\, \sqrt{\hat{B}_+\hat{T}^\dagger} + \hat{H}_1^{1/4}\, \sin(\hat{\omega}_2 t)\, \sin(\hat{\omega}_1 t)\, \hat{H}_2^{1/4} \right\} \tag{12c}$$
$$\hat{F}_{22}(t) = \gamma \left\{ \hat{C}^\dagger\, \sin(\hat{\omega}_1 t)\, \hat{T}\hat{B}_- \, \cos(\hat{\omega}_2 t) + \cos(\hat{\omega}_2 t)\, \hat{B}_+\hat{T}^\dagger \, \sin(\hat{\omega}_1 t)\, \hat{C} \right\}$$
$$= i\gamma \left\{ \sqrt{\hat{B}_+\hat{T}^\dagger}\, \cos(\hat{\omega}_1 t)\, \sin(\hat{\omega}_2 t)\, \hat{H}_1^{1/4} - \hat{H}_1^{1/4}\, \sin(\hat{\omega}_2 t)\, \cos(\hat{\omega}_1 t)\, \sqrt{\hat{T}\hat{B}_-} \right\}, \tag{12d}$$

where $\gamma = 4\alpha^2\beta/\hbar^2$. Here we used the properties (1), (4) and (7), together with the following operators relations

$$\hat{C}\, \sqrt{\hat{B}_+\hat{T}^\dagger} = -\sqrt{\hat{T}\hat{B}_-}\, \hat{C}^\dagger = i\hat{H}_2^{1/4} \tag{13a}$$
$$\sqrt{\hat{B}_+\hat{T}^\dagger}\, \hat{C} = -\hat{C}^\dagger\, \sqrt{\hat{T}\hat{B}_-} = i\hat{H}_1^{1/4}. \tag{13b}$$

Now, keeping in mind that $[\hat{\nu}_j, \hat{\omega}_j] = 0$, $(j = 1$, or $2)$, so we may use the trigonometric relationships involving products of trigonometric functions with arguments $\hat{\nu}_j t$ and $\hat{\omega}_j t$ (since we have $\exp(\hat{\nu}_j t)\, \exp(\pm\hat{\omega}_j t) = \exp[(\hat{\nu}_j \pm \hat{\omega}_j)t]$). Then, using those relationships, the following commutators

$$\left[\hat{\nu}_1, \hat{H}_2\right] = \left[\hat{\omega}_1, \hat{H}_2\right] = \left[\hat{\nu}_2, \hat{H}_1\right] = \left[\hat{\omega}_2, \hat{H}_1\right] = 0\,, \tag{14}$$

and the same properties (1), (4) and (7), we can show that

$$\hat{y}_1(t)\, \hat{F}_{11}(t) = i\frac{\gamma}{2}\sqrt{\hat{T}\hat{B}_-}\, \{\cos[(\hat{\nu}_2 - \hat{\omega}_2)t]\, \sin(\hat{\omega}_1 t) + \cos[(\hat{\nu}_2 + \hat{\omega}_2)t]\, \sin(\hat{\omega}_1 t)\}\, \hat{H}_2^{1/4}$$
$$+ i\frac{\gamma}{2}\hat{H}_2^{1/4}\, \{\sin[(\hat{\nu}_1 - \hat{\omega}_1)t]\, \cos(\hat{\omega}_2 t) - \sin[(\hat{\nu}_1 + \hat{\omega}_1)t]\, \cos(\hat{\omega}_2 t)\}\, \sqrt{\hat{B}_+\hat{T}^\dagger} \tag{15a}$$
$$\hat{y}_1(t)\, \hat{F}_{12}(t) = \frac{\gamma}{2}\sqrt{\hat{T}\hat{B}_-}\, \{\cos[(\hat{\nu}_2 - \hat{\omega}_2)t]\, \cos(\hat{\omega}_1 t) + \cos[(\hat{\nu}_2 + \hat{\omega}_2)t]\, \cos(\hat{\omega}_1 t)\}\, \sqrt{\hat{T}\hat{B}_-}$$
$$+ \frac{\gamma}{2}\hat{H}_2^{1/4}\, \{\sin[(\hat{\nu}_1 + \hat{\omega}_1)t]\, \sin(\hat{\omega}_2 t) - \sin[(\hat{\nu}_1 - \hat{\omega}_1)t]\, \sin(\hat{\omega}_2 t)\}\, \hat{H}_1^{1/4} \tag{15b}$$
$$\hat{y}_2(t)\, \hat{F}_{21}(t) = \frac{\gamma}{2}\sqrt{\hat{B}_+\hat{T}^\dagger}\, \{\cos[(\hat{\nu}_1 - \hat{\omega}_1)t]\, \cos(\hat{\omega}_2 t) + \cos[(\hat{\nu}_1 + \hat{\omega}_1)t]\, \cos(\hat{\omega}_2 t)\}\, \sqrt{\hat{B}_+\hat{T}^\dagger}$$
$$+ \frac{\gamma}{2}\hat{H}_1^{1/4}\, \{\sin[(\hat{\nu}_2 + \hat{\omega}_2)t]\, \sin(\hat{\omega}_1 t) - \sin[(\hat{\nu}_2 - \hat{\omega}_2)t]\, \sin(\hat{\omega}_1 t)\}\, \hat{H}_2^{1/4} \tag{15c}$$
$$\hat{y}_2(t)\, \hat{F}_{22}(t) = i\frac{\gamma}{2}\sqrt{\hat{B}_+\hat{T}^\dagger}\, \{\cos[(\hat{\nu}_1 - \hat{\omega}_1)t]\, \sin(\hat{\omega}_2 t) + \cos[(\hat{\nu}_1 + \hat{\omega}_1)t]\, \sin(\hat{\omega}_2 t)\}\, \hat{H}_1^{1/4}$$
$$+ i\frac{\gamma}{2}\hat{H}_1^{1/4}\, \{\sin[(\hat{\nu}_2 - \hat{\omega}_2)t]\, \cos(\hat{\omega}_1 t) - \sin[(\hat{\nu}_2 + \hat{\omega}_2)t]\, \cos(\hat{\omega}_1 t)\}\, \sqrt{\hat{T}\hat{B}_-}\,. \tag{15d}$$

In a similar way, we can show that

$$
\begin{aligned}
\hat{z}_1(t)\,\hat{F}_{11}(t) &= i\frac{\gamma}{2}\sqrt{\hat{T}\hat{B}_-}\,\{\sin[(\hat{\nu}_2-\hat{\omega}_2)t]\,\sin(\hat{\omega}_1 t)+\sin[(\hat{\nu}_2+\hat{\omega}_2)t]\,\sin(\hat{\omega}_1 t)\}\,\hat{H}_2^{1/4} \\
&\quad - i\frac{\gamma}{2}\hat{H}_2^{1/4}\,\{\cos[(\hat{\nu}_1-\hat{\omega}_1)t]\,\cos(\hat{\omega}_2 t)-\cos[(\hat{\nu}_1+\hat{\omega}_1)t]\,\cos(\hat{\omega}_2 t)\}\,\sqrt{\hat{B}_+\hat{T}^\dagger} \qquad (16a) \\
\hat{z}_1(t)\,\hat{F}_{12}(t) &= \frac{\gamma}{2}\sqrt{\hat{T}\hat{B}_-}\,\{\sin[(\hat{\nu}_2-\hat{\omega}_2)t]\,\cos(\hat{\omega}_1 t)+\sin[(\hat{\nu}_2+\hat{\omega}_2)t]\,\cos(\hat{\omega}_1 t)\}\,\sqrt{\hat{T}\hat{B}_-} \\
&\quad - \frac{\gamma}{2}\hat{H}_2^{1/4}\,\{\cos[(\hat{\nu}_1+\hat{\omega}_1)t]\,\sin(\hat{\omega}_2 t)-\cos[(\hat{\nu}_1-\hat{\omega}_1)t]\,\sin(\hat{\omega}_2 t)\}\,\hat{H}_1^{1/4} \qquad (16b) \\
\hat{z}_2(t)\,\hat{F}_{21}(t) &= \frac{\gamma}{2}\sqrt{\hat{B}_+\hat{T}^\dagger}\,\{\sin[(\hat{\nu}_1-\hat{\omega}_1)t]\,\cos(\hat{\omega}_2 t)+\sin[(\hat{\nu}_1+\hat{\omega}_1)t]\,\cos(\hat{\omega}_2 t)\}\,\sqrt{\hat{B}_+\hat{T}^\dagger} \\
&\quad - \frac{\gamma}{2}\hat{H}_1^{1/4}\,\{\cos[(\hat{\nu}_2+\hat{\omega}_2)t]\,\sin(\hat{\omega}_1 t)-\cos[(\hat{\nu}_2-\hat{\omega}_2)t]\,\sin(\hat{\omega}_1 t)\}\,\hat{H}_2^{1/4} \qquad (16c) \\
\hat{z}_2(t)\,\hat{F}_{22}(t) &= i\frac{\gamma}{2}\sqrt{\hat{B}_+\hat{T}^\dagger}\,\{\sin[(\hat{\nu}_1-\hat{\omega}_1)t]\,\sin(\hat{\omega}_2 t)+\sin[(\hat{\nu}_1+\hat{\omega}_1)t]\,\sin(\hat{\omega}_2 t)\}\,\hat{H}_1^{1/4} \\
&\quad - i\frac{\gamma}{2}\hat{H}_1^{1/4}\,\{\cos[(\hat{\nu}_2-\hat{\omega}_2)t]\,\cos(\hat{\omega}_1 t)-\cos[(\hat{\nu}_2+\hat{\omega}_2)t]\,\cos(\hat{\omega}_1 t)\}\,\sqrt{\hat{T}\hat{B}_-}\,. \qquad (16d)
\end{aligned}
$$

The non-commutativity between the operators $\hat{\omega}_1$ and $\hat{\omega}_2$ imply that to calculate the integrals involving the terms given by Eqs. (15) and (16) we need to use the series expansion of the trigonometric functions. In this case the integrals can be easily done because the time variable can be considered as a parameter factor. Finally, using these results into Eq. (3.30) is trivial to find the expression (3.31) for the matrix elements of the particular solution.

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